

# Algebra, Number theory and Combinatorics

## Individual Contest

**Question 1.** Let  $M_n$  be the space of real  $n \times n$  matrices with euclidean topology. Let  $G = GL_n(\mathbb{R})$  act on  $M_n$  by conjugation. For a diagonal matrix  $D \in M_n$ , denote by  $G_D$  the  $G$ -orbit of  $D$ . Show that  $G_D$  is closed in  $M_n$ .

**Question 2.** Let  $\mathcal{A}$  be the abelian category of finite abelian groups. Show that  $\mathcal{A}$  has neither enough projective objects nor enough injective objects.

**Question 3.** Let  $G$  be a group of cardinality  $n$ . Prove that  $G$  has at most  $n^{\log_2 n}$  automorphisms.

**Question 4.** Let  $K$  be a field, we define  $l(K)$  to be the smallest natural number  $n$  such that there are  $n$  elements  $x_1, \dots, x_n$  in  $K$  satisfying

$$-1 = x_1^2 + x_2^2 + \dots + x_n^2.$$

If no such  $n$  exists, we denote  $l(K)$  by  $\infty$ .

1. Prove that

$$l(K) = l(K(t)),$$

here  $K(t)$  is the field of rational functions with coefficients in  $K$ .

2. Compute  $l(K)$  when  $K$  are the following fields:  $\mathbb{Q}(e^{2i\pi/3})$ ,  $\mathbb{Q}(e^{2i\pi/3}2^{1/3})$ .